1 Investigation of Right-angled Triangles

Answering this investigation

Some tasks in this investigation require you to use Geogebra (www.geogebra.org). All the files are viewable online, so you will not need to install anything. If you follow the links using a modern internet browser, you should have no problems. This does mean that you might have problems using a phone or an old computer (but the school computers should be fine).

All your answers should be written on this investigation in pencil.

Some stuff to know at the start

There are some important concepts and terms that you will need for this investigation:

Right-angled triangle. This is a triangle with one angle of 90°.

Hypotenuse. This is the longest side in a right-angled triangle. Note that it is always the side opposite the right-angle.

Similar triangles. There are two ways to know that two triangles are similar:

1. All corresponding angles in the triangles (i.e. angles in the same place in each triangle) are equal.
2. The sides of the triangles are in the same proportions.

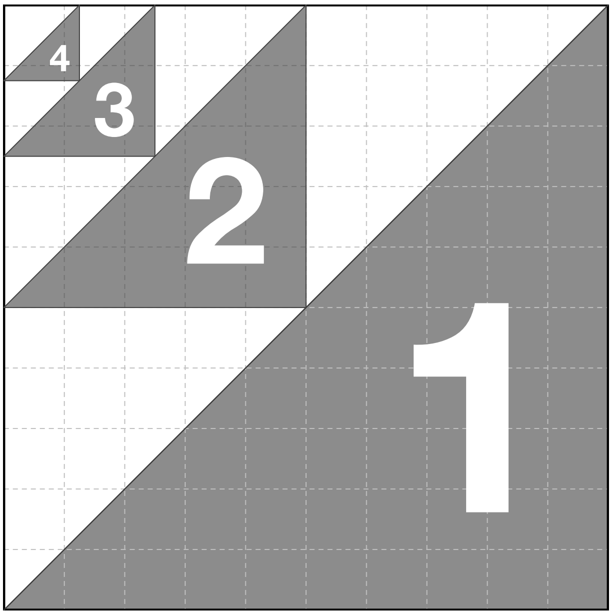
Area of a rectangle. The area of a rectangle is its length times its breadth. A square is a special type of rectangle where the length and breadth are equal.

Area of a right-angled triangle. The area is $\frac{1}{2} \times \text{base} \times \text{height}$. (It is not hard to see that a right-angled triangle can be "doubled" by placing an exact copy of it next to it to form a rectangle.)

Theorem. In Maths, a theorem is a statement that can be *proven* to be true. Don't confuse a theorem with a theory – a theory is an idea that hasn't been proved completely.

TASK 1: Similar triangles

Four triangles are drawn inside a 10-unit by 10-unit square, as shown below.



Answer the following questions in the spaces provided:

1.1	What fraction of the area of the square is triangle 1?	
1.2	What fraction of triangle 1 is triangle 2?	
1.3	What fraction of the square is triangle 2?	
1.4	How many times can triangle 3 fit into triangle 2?	
1.5	How many times can triangle 2 fit into triangle 1?	
1.6	What fraction of the area of triangle 1 is triangle 3?	
1.7	What fraction of the square is triangle 3?	

1.8 How many times can triangle 4 fit into triangle 3?

1.9 How many times can triangle 4 fit into triangle 2?

1.10 How many times can triangle 4 fit into triangle 1?

1.11 How many times can triangle 4 fit into the square?

You will have noticed that all four triangles are the same shape. Their corresponding angles are the same, but the lengths of the corresponding sides differ. Because their corresponding angles are equal, we call them *similar* triangles.

From experience, people discovered that a triangle with its sides in the ratio 3 : 4 : 5 is a right-angled triangle. It turns out that if we have a triangle with its sides in a ratio of 6 : 8 : 10, it's also a right-angled triangle. If we simplify the ratio, we get 3 : 4 : 5. This means that any triangle that is *similar to* a 3 : 4 : 5 triangle is also a right-angled triangle, because its sides will be in the same proportions.

Because 3 : 4 : 5 is special in this way, we call it a *Pythagorean triple*. It is not the only triple, and you will find some others in the next task.

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TASK 2: Triples

For this task, you are going to find *three* other triangles that have whole-number lengths and are also right-angled triangles. In other words, you are going to find some different Pythagorean triples.

Follow the steps below and fill in the table that follows:

1. Open https://www.geogebratube.org/student/m138710.
2. By moving the points labelled A and B in the diagram, find three different right-angled triangles where all three side lengths are whole numbers. Two of your triangles MAY NOT be $3:4:5$ triangles, but must be in a different ratio.
3. In the table below, record the *lengths* of the sides for each of the right-angled triangles that you find. Remember that all three lengths must be whole numbers.
4. Once you have calculated the lengths of the sides, calculate the *ratio* of the sides to one another, and write the fully simplified ratio in the table.

	Length of red side	Length of blue side	Length of Hypotenuse	Ratio of sides
Example:	3	4	5	$3:4:5$
Triangle 1:				
Triangle 2:				
Triangle 3:				

When ancient civilisations discovered these special right-angled triangles, they did it by observation of the physical world. From these, they were able to notice a very important and useful pattern.

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TASK 3: Noticing the pattern

For this task, you will investigate a number of different right-angled triangles using GeoGebra to look for a pattern. Answer the questions in the spaces provided.

Open https://www.geogebratube.org/student/m138697.

3.1 By moving the blue points in the Geogebra file, create different right-angled triangles to help complete the following table. Note that a, b, and c are lengths of sides, and are labelled on the diagram. Side-lengths MUST be whole numbers for this task.

	a	b	c	a^2	b^2	c^2
Example:	3	4	5	9	16	25
Triangle 1:						
Triangle 2:						
Triangle 3:						
Triangle 4:						
Triangle 5:						

(10)

3.2 Describe (in words) a relationship that you notice is true for all five triangles. (*If you struggle to find a relationship, consider asking someone or looking online.*)

(3)

3.3 Write down the relationship using symbols and mathematical notation:

(3)

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2 The Theorem of Pythagoras

In the previous task, you should have discovered the following relationship, which we call the Theorem of Pythagoras:

> **Theorem of Pythagoras:** In a right-angled triangle with sides a, b, and c, where c is the hypotenuse, then:
>
> $$a^2 + b^2 = c^2.$$

To know that this relationship is true mathematically, we have to prove that it is *always* true for *any* right-angled triangle.

There are many different ways to prove this theorem. We will only look at two of them (although one book on the topic has 370 *different* proofs for the theorem!).

TASK 4: Pythagoras's Proof

Pythagoras definitely wasn't the first person to know of the theorem, but he was probably the first person known to have proved it.

We will first look at the proof using a triangle with known measurements, and then we will use algebra.

4.1 The following two diagrams show Pythagoras's idea of a proof by rearrangement:

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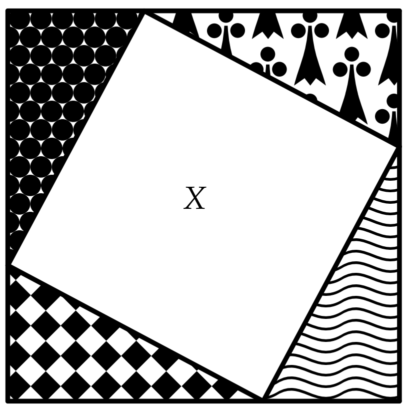

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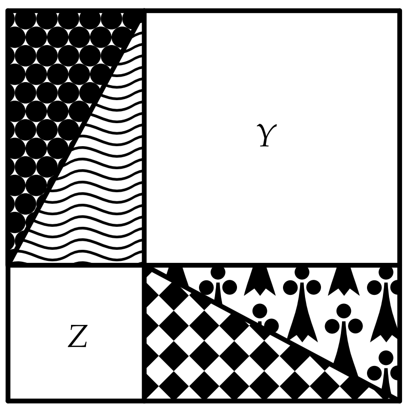

These two diagrams show a couple of different ways of rearranging a right-angled triangle with a side of 8 units and a side of 15 units. Note that while the triangles are patterned differently, this is to help you see how they have been rearranged – they are actually all copies of the same triangle.

In the space provided, answer the following questions:

4.1.1 What is the total side-length of the square diagram on the left?

4.1.2 What is the total area of the diagram on the left?

4.1.3 What is the total side-length of the square diagram on the right?

4.1.4 What is the total area of the diagram on the right?

4.1.5 In the square on the right, there are two smaller white squares, Y and Z. What is the area of Z?

4.1.6 What is the area of Y?

4.1.7 What is the area of *one* of the patterned right-angled triangles?

4.1.8 Hence calculate the area of all the patterned triangles in the diagram on the left.

4.1.9 Calculate the area of all the patterned triangles in the diagram on the right.

We can calculate the area of the white, unpatterned squares by subtracting the areas of the patterned triangles from the total areas of the squares.

4.1.10 Hence calculate the *combined* area of the two white squares Y and Z in the right-hand diagram.

4.1.11 What is the area of the white square X in the diagram on the left? Give a reason for your answer.

4.1.12 One side of white square X is the hypotenuse of the patterned triangle. Calculate the length of the hypotenuse.

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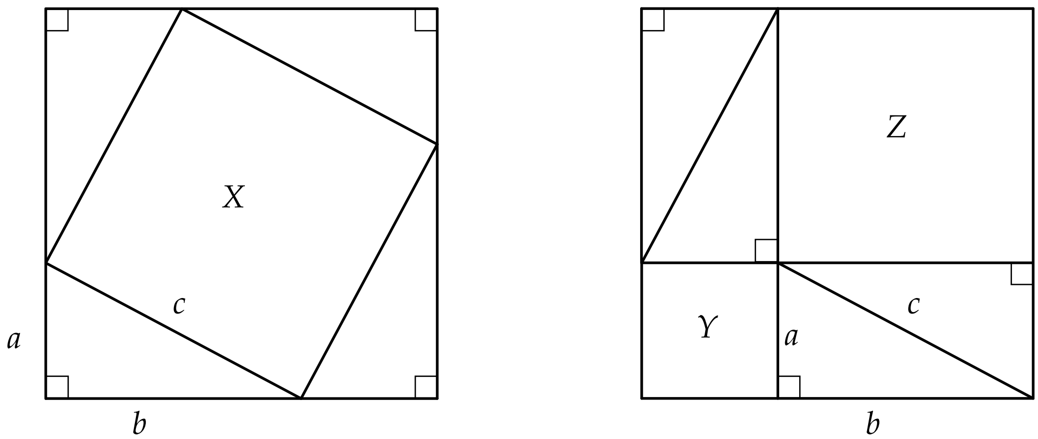
By doing the above, we can see that there may be a general method for finding the hypotenuse for any right-angled triangle.

Please turn over

4.2 The following diagrams represent a *general* arrangement of a right-angled triangle with sides of length a and b, and a hypotenuse of length c. Note that this set of diagrams could be for *any* right-angled triangle, and is not limited to a specific case.



Answer the following in the spaces provided. Note that answers will be algebraic and not numeric.

4.2.1 What is the total side-length of the square diagram on the left?

4.2.2 What is the total area of the diagram on the left?

4.2.3 What is the total side-length of the square diagram on the right?

4.2.4 What is the total area of the diagram on the right?

4.2.5 In the square on the right, there are two smaller squares, Y and Z. What is the area of Z?

4.2.6 What is the area of Y?

4.2.7 What is the area of *one* of right-angled triangles used to make up these diagrams?

4.2.8 Hence calculate the area of all four right-angled triangles in the diagram on the left.

4.2.9 Calculate the area of all four right-angled triangles in the diagram on the right.

4.2.10 Hence calculate the *combined* area of the two squares Y and Z in the right-hand diagram.

4.2.11 Calculate the area of X in terms of c.

4.2.12 Without performing a calculation, what is the area of the square X in terms of a and b? Give a reason for your answer.

4.2.13 What is the relationship between the area of square X and squares Y and Z? Give an algebraic equation that supports your answer. (This proves the theorem algebraically!)

4.2.14 One side of the square X is the hypotenuse of the right-angled triangle with sides a, b, and c. Calculate the length of the hypotenuse in terms of a and b.

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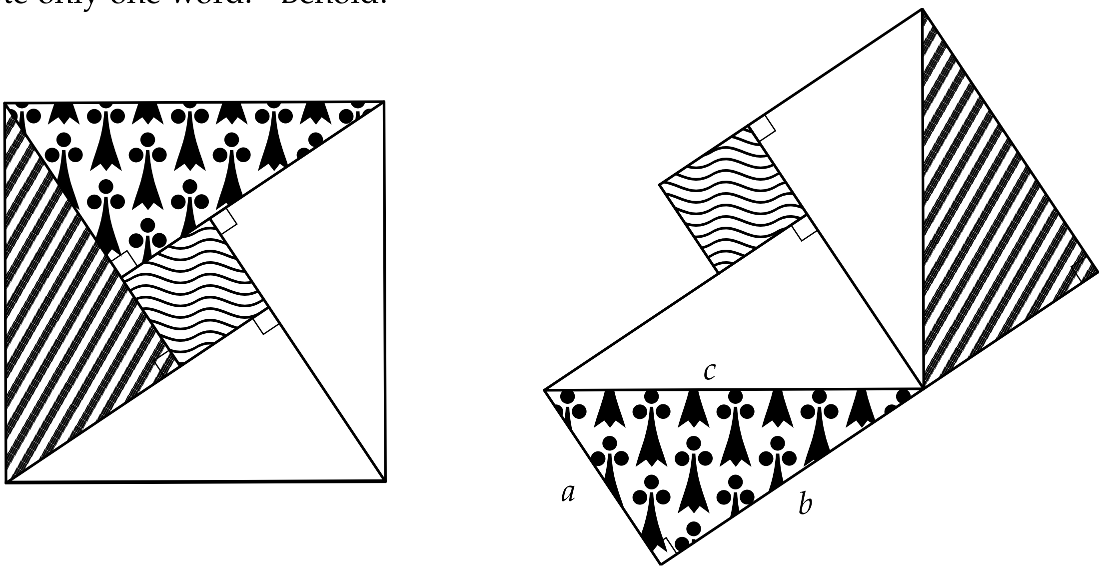
TASK 5: Behold!

Bhāskara, a 12th-century Indian Mathematician, included this proof in *Lilavati*, a book he wrote on Mathematics. Along with the diagrams shown below, he wrote only one word: "Behold!"

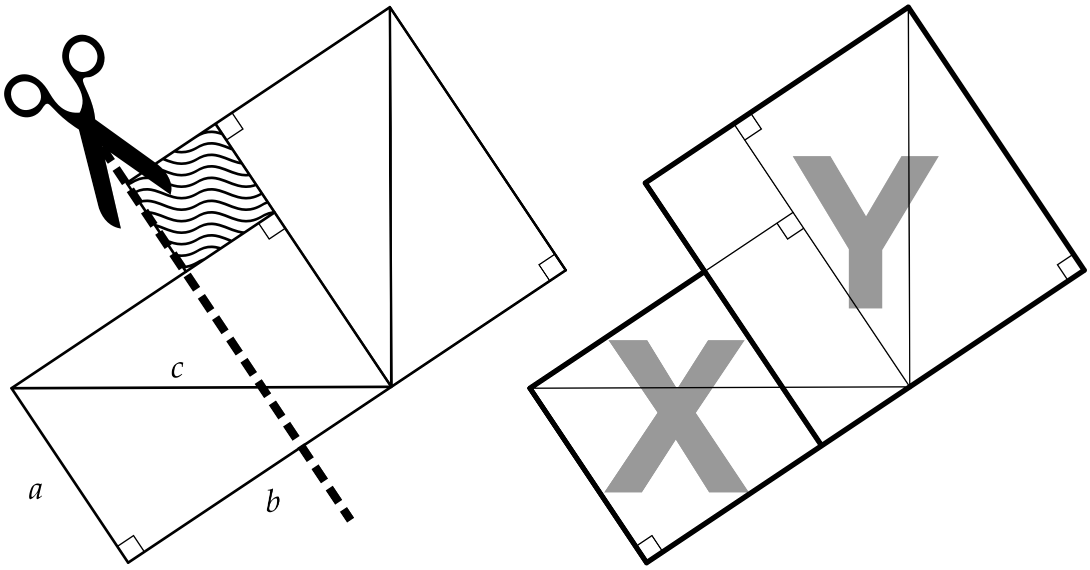


The two diagrams are both made up of four copies of a right-angled triangle with sides a, b, and c. To get from the diagram on the left to the one on the right, the two shaded triangles on the top-left are moved to new positions. To understand how the diagram on the right is formed, go to http://britton.disted.camosun.bc.ca/geometry/behold.html.

5.1 What is the area of the diagram on the left? (*Hint: it is a square.*)

5.2 What is the total area of the diagram on the right? Do not perform any calculations, but give a reason for your answer.

We now take the right-most diagram, and consider it a little differently by imagining it as being made up of two squares, labelled X and Y:



5.3 What is the side-length of the square X, and what is its area?

5.4 What is the side-length of the square Y and what is its area?

5.5 Hence find the combined area of squares X and Y, in terms of a and b.

5.6 Using your solutions to 5.2 and 5.5, write down an equation showing the relationship between a, b, and c.

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TOTAL: 60